Compatibility of Local and Global Langlands Correspondences

COMPATIBILITY OF LOCAL AND GLOBAL LANGLANDS CORRESPONDENCES

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Abstract. We prove the compatibility of local and global Langlands correspondences for GL_n, which was proved up to semisimplification in [HT]. More precisely, for the n-dimensional l-adic representation $R_l(\Pi)$ of the Galois group of a CM-field L attached to a conjugate self-dual regular algebraic cuspidal automorphic representation Π, which is square integrable at some finite place, we show that Frobenius semisimplification of the restriction of $R_l(\Pi)$ to the decomposition group of a prime v of L not dividing l corresponds to Π_v by the local Langlands correspondence.

Introduction

This paper is a continuation of [HT]. Let L be an (imaginary) CM field and let Π be a regular algebraic cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ which is conjugate self-dual ($\Pi \circ c \cong \Pi^\vee$) and square integrable at some finite place. In [HT] it is explained how to attach to Π and an arbitrary rational prime l (and an isomorphism $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$) a continuous semisimple representation

$$R_l(\Pi) : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

which is characterised as follows. For every finite place v of L not dividing l

$$\imath R_l(\Pi)|_{W_{L_v}}^{\mathrm{ss}} = \mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)^{\mathrm{ss}},$$

where rec denotes the local Langlands correspondence and ss denotes the semisimplification (see [HT] for details). In that book it is also shown that Π_v is tempered for all finite places v.

In this paper we strengthen this result to completely identify $R_l(\Pi)|_{I_v}$ for $v \nmid l$. In particular, we prove the following theorem.

Theorem A. *If $v \nmid l$ then the Frobenius semisimplification of $R_l(\Pi)|_{W_{L_v}}$ is the l-adic representation attached to $\imath^{-1}\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$.*

As $R_l(\Pi)$ is semisimple and $\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$ is indecomposable if Π_v is square integrable, we obtain the following corollary.

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Corollary B. *If Π_v is square integrable at a finite place $v \not| l$, the representation $R_l(\Pi)$ is irreducible.*

Using base change it is easy to reduce to the case that Π_v has an Iwahori fixed vector. We descend Π to an automorphic representation π of a unitary group G which locally at v looks like GL_n and at infinity looks like $U(n-1,1) \times U(n,0)^{[L:\mathbb{Q}]/2-1}$. Then we realise $R_l(\Pi)$ in the cohomology of a Shimura variety X associated to G with Iwahori level structure at v. More precisely, for some l-adic sheaf $\mathcal{L}$, the π^p-isotypic component of $H^{n-1}(X, \mathcal{L})$ is, up to semisimplification and some twist, $R_l(\Pi)^a$ (for some $a \in \mathbb{Z}_{>0}$). We show that X has semistable reduction and use the results of [HT] to calculate the cohomology of the (smooth, projective) strata of the reduction of X above p as a virtual $G(\mathbb{A}^{\infty,p}) \times F^{\mathbb{Z}}$-module (where F denotes Frobenius). This description and the temperedness of Π_v shows that the π^p-isotypic component of the cohomology of any strata is concentrated in the middle dimension. This implies that the π^p-isotypic component of the Rapoport-Zink weight spectral sequence degenerates at E_1, which allows us to calculate the action of inertia at v on $H^{n-1}(X, \mathcal{L})$.

In the special case that Π_v is a twist of a Steinberg representation and Π_∞ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I].

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1. The main theorem

We write F^{ac} for an algebraic closure of a field F. Let l be a rational prime and fix an isomorphism $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$.

Suppose that $p \neq l$ is another rational prime. Let $K/\mathbb{Q}_p$ be a finite extension. We will let $\mathcal{O}_K$ denote the ring of integers of K, $\wp_K$ the unique maximal ideal of $\mathcal{O}_K$, v_K the canonical valuation $K^\times \twoheadrightarrow \mathbb{Z}$, $k(v_K)$ the residue field $\mathcal{O}_K/\wp_K$ and $|\ |_K$ the absolute value normalised by $|x|_K = (\# k(v_K))^{-v_K(x)}$. We will let Frob_{v_K} denote the geometric Frobenius element of $\mathrm{Gal}\,(k(v_K)^{ac}/k(v_K))$. We will let I_{v_K} denote the kernel of the natural surjection $\mathrm{Gal}\,(K^{ac}/K) \twoheadrightarrow \mathrm{Gal}\,(k(v_K)^{ac}/k(v_K))$. We will let W_K denote the preimage under $\mathrm{Gal}\,(K^{ac}/K) \twoheadrightarrow \mathrm{Gal}\,(k(v_K)^{ac}/k(v_K))$ of $\mathrm{Frob}_{v_K}^{\mathbb{Z}}$ endowed with a topology by decreeing that I_K with its usual topology is an open subgroup of W_K. Local class field theory provides a canonical isomorphism $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$, which takes uniformisers to lifts of Frob_{v_K}.

Let Ω be an algebraically closed field of characteristic 0 and of the same cardinality as $\mathbb{C}$. (Thus in fact $\Omega \cong \mathbb{C}$.) By a *Weil-Deligne representation* of W_K over Ω we mean a finite dimensional Ω-vector space V together with a homomorphism $r : W_K \to GL(V)$ with open kernel and an element $N \in \mathrm{End}\,(V)$ which satisfies

$$r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_K^{-1}(\sigma)|_K N.$$

We sometimes denote a Weil-Deligne representation by (V, r, N) or simply (r, N).

We call (V, r, N) *Frobenius semisimple* if r is semisimple. If (V, r, N) is any Weil-Deligne representation we define its *Frobenius semisimplification* $(V, r, N)^{F\text{-ss}} = (V, r^{\mathrm{ss}}, N)$ as follows. Choose a lift ϕ of Frob_{v_K} to W_K. Let $r(\phi) = su = us$ where $s \in GL(V)$ is semisimple and $u \in GL(V)$ is unipotent. For $n \in \mathbb{Z}$ and $\sigma \in I_K$ set $r^{\mathrm{ss}}(\phi^n \sigma) = s^n r(\sigma)$. This is independent of the choices, and gives a Frobenius semisimple Weil-Deligne representation.

One of the main results of [HT] is that, given a choice of $(\# k(v_K))^{1/2} \in \Omega$, there is a bijection rec (the local Langlands correspondence) from isomorphism classes of irreducible smooth representations of $GL_n(K)$ over Ω to isomorphism classes of n-dimensional Frobenius semisimple Weil-Deligne representations of W_K, and that this bijection is natural in a number of respects. (See [HT] for details.)

We will call a Weil-Deligne representation of W_K over $\mathbb{Q}_l^{ac}$ *bounded* if for some (and hence all) $\sigma \in W_K - I_K$ all the eigenvalues of $r(\sigma)$ are l-adic units. There is an equivalence of categories between bounded Weil-Deligne representations of W_K over $\mathbb{Q}_l^{ac}$ and continuous representations of $\mathrm{Gal}\,(K^{ac}/K)$ on finite dimensional $\mathbb{Q}_l^{ac}$-vector spaces as follows. Fix a lift $\phi \in W_K$ of Frob_{v_K} and a continuous homomorphism $t : I_K \twoheadrightarrow \mathbb{Z}_l$. Send a Weil-Deligne representation (V, r, N) to (V, ρ), where ρ is the unique continuous representation of $\mathrm{Gal}\,(K^{ac}/K)$ on V such that

$$\rho(\phi^n \sigma) = r(\phi^n \sigma) \exp\bigl(t(\sigma) N\bigr)$$

for all $n \in \mathbb{Z}$ and $\sigma \in I_K$. Up to natural isomorphism this functor is independent of the choices of t and ϕ. We will write $\mathrm{WD}(V, \rho)$ for the Weil-Deligne representation corresponding to a continuous representation (V, ρ). If $\mathrm{WD}(V, \rho) = (V, r, N)$, then have $\rho|_{W_K}^{\mathrm{ss}} \cong r^{\mathrm{ss}}$. (See [T], §4 and [D], §8 for details.)

Now suppose that L is a finite, imaginary CM extension of $\mathbb{Q}$. Let $c \in \mathrm{Aut}\,(L)$ denote complex conjugation. Suppose that Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- Π_∞ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from L to $\mathbb{Q}$ of GL_n;
- and for some finite place x of L the representation Π_x is square integrable.

(In this paper 'square integrable' (resp. 'tempered') will mean the twist by a character of a pre-unitary representation which is square integrable (resp. tempered).) In [HT] (see theorem C in the introduction of [HT]) it is shown that there is a unique continuous semisimple representation

$$R_l(\Pi) : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

such that for each finite place $v \not| l$ of L

$$\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big) = (\imath R_l(\Pi)|^{\mathrm{ss}}_{W_{L_v}}, N)$$

for some N. Moreover it is shown that Π_v is tempered for all finite places v of L, which completely determines the N (see lemma 1.3 below). If $n=1$ both these assertions are true without the assumption that $\Pi \circ c \cong \Pi^\vee$.

The main theorem of this paper identifies the N of $\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})$ with the above N. More precisely we prove the following.

Theorem 1.1. *Keep the above notation and assumptions. Then for each finite place $v \not| l$ of L there is an isomorphism*

$$\imath \mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{F\text{-ss}} \cong \mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$$

of Weil-Deligne representations over $\mathbb{C}$.

As $R_l(\Pi)$ is semisimple and $\mathrm{rec}\big(\Pi_v^\vee |\det|^{\frac{1-n}{2}}\big)$ is indecomposable if Π_v is square integrable, we have the following corollary.

Corollary 1.2. *If Π_v is square integrable for a finite place $v \not| l$, then the representation $R_l(\Pi)$ is irreducible.*

In the rest of this section we consider some generalities on Galois representations and Weil-Deligne representations. First consider Weil-Deligne representations over an algebraically closed field Ω of characteristic zero and the same cardinality as $\mathbb{C}$. For a finite extension K'/K of p-adic fields, we define

$$(V, r, N)|_{W_{K'}} = (V, r|_{W_{K'}}, N).$$

If (W, r) is a finite dimensional representation of W_K with open kernel and if $s \in \mathbb{Z}_{\geq 1}$ we will write $\mathrm{Sp}_s(W)$ for the Weil-Deligne representation

$$\big(W^s,\ r|\mathrm{Art}_K^{-1}|_K^{s-1} \oplus \cdots \oplus r|\mathrm{Art}_K^{-1}|_K \oplus r,\ N\big)$$

where $N : r|\mathrm{Art}_K^{-1}|_K^{i-1} \xrightarrow{\sim} r|\mathrm{Art}_K^{-1}|_K^{i}$ for $i = 1, ..., s-1$. This defines $\mathrm{Sp}_s(W)$ uniquely (up to isomorphism). If W is irreducible then $\mathrm{Sp}_s(W)$ is indecomposable and every indecomposable Weil-Deligne representation is of the form $\mathrm{Sp}_s(W)$ for a unique s and a unique irreducible W. If π is an irreducible cuspidal representation of $GL_g(K)$ then $\mathrm{rec}(\pi) = (r, 0)$ with r irreducible. Moreover for any $s \in \mathbb{Z}_{\geq 1}$ we have (in the notation of section I.3 of [HT]) $\mathrm{rec}(\mathrm{Sp}_s(\pi)) = \mathrm{Sp}_s(r)$.

If $q \in \mathbb{R}_{>0}$, then by a *Weil q-number* we mean $\alpha \in \mathbb{Q}^{ac}$ such that for all $\sigma : \mathbb{Q}^{ac} \hookrightarrow \mathbb{C}$ we have $(\sigma\alpha)(c\sigma\alpha) = q$. We will call a Weil-Deligne representation (V, r, N) of W_K *strictly pure of weight* $k \in \mathbb{R}$ if for some (and hence every) lift ϕ of Frob_{v_K}, every eigenvalue α of $r(\phi)$ is a Weil $(\# k(v_K))^k$-number. In this case we must have $N = 0$. We will call (V, r, N) *mixed* if it has an increasing filtration Fil^W_i with $\mathrm{Fil}^W_i V = V$ for $i >> 0$ and $= (0)$ for

$i << 0$, such that the i-th graded piece is strictly pure of weight i. If (V,r,N) is mixed then there is a unique choice of filtration Fil_i^W, and $N(\mathrm{Fil}_i^W V) \subset \mathrm{Fil}_{i-2}^W V$. Finally we will call (V,r,N) *pure of weight* k if it is mixed with all weights in $k+\mathbb{Z}$ and if for all $i \in \mathbb{Z}_{>0}$

$$N^i : \mathrm{gr}_{k+i}^W V \xrightarrow{\sim} \mathrm{gr}_{k-i}^W V.$$

If W is strictly pure of weight k, then $\mathrm{Sp}_s(W)$ is pure of weight $k-(s-1)$ for any $s \in \mathbb{Z}_{\geq 1}$. (It is generally conjectured that if X is a proper smooth variety over a p-adic field K, then $\mathrm{WD}\big(H^i(X \times_K K^{ac}, \mathbb{Q}_l^{ac})\big)$ is pure of weight i in the above sense.)

Lemma 1.3. (1) *(V,r,N) is pure if and only if $(V,r,N)^{F\text{-ss}}$ is.*

(2) *If L/K is a finite extension, then (V,r,N) is pure if and only if $(V,r,N)|_{W_L}$ is pure.*

(3) *An irreducible smooth representation π of $GL_n(K)$ has $\sigma\pi$ tempered for all $\sigma : \Omega \hookrightarrow \mathbb{C}$ if and only if $\mathrm{rec}(\pi)$ is pure of some weight.*

(4) *Given (V,r) with r semisimple, there is, up to equivalence, at most one choice of N which makes (V,r,N) pure.*

(5) *If (V,r,N) is a Frobenius semisimple Weil-Deligne representation which is pure of weight k and if $W \subset V$ is a Weil-Deligne subrepresentation, then the following are equivalent:*
   (a) *$\bigwedge^{\dim W} W$ is pure of weight $k \dim W$,*
   (b) *W is pure of weight k,*
   (c) *W is a direct summand of V.*

(6) *Suppose that (V,r,N) is a Frobenius semisimple Weil-Deligne representation which is pure of weight k. Suppose also that $\mathrm{Fil}^j V$ is a decreasing filtration of V by Weil-Deligne subrepresentations such that $\mathrm{Fil}^j V = (0)$ for $j >> 0$ and $\mathrm{Fil}^j V = V$ for $j << 0$. If for each j*

$$\bigwedge\nolimits^{\dim \mathrm{gr}^j V} \mathrm{gr}^j V$$

is pure of weight $k \dim \mathrm{gr}^j V$, then

$$V \cong \bigoplus_j \mathrm{gr}^j V$$

and each $\mathrm{gr}^j V$ is pure of weight k.

Proof: The first two parts are straightforward (using the fact that the filtration Fil_i^W is unique). For the third part recall that an irreducible smooth representation $\mathrm{Sp}_{s_1}(\pi_1) \boxplus \cdots \boxplus \mathrm{Sp}_{s_t}(\pi_t)$ (see section I.3 of [HT]) is tempered if and only if the absolute values of the central characters of the $\mathrm{Sp}_{s_i}(\pi_i)$ are all equal.

Suppose that (V,r,N) is Frobenius semisimple and pure of weight k. As a W_K-module we can write uniquely $V = \oplus_{i \in \mathbb{Z}} V_i$ where $(V_i, r, 0)$ is strictly pure of weight $k+i$. For $i \in \mathbb{Z}_{\geq 0}$ let $V(i)$ denote the kernel of $N^{i+1} : V_i \to V_{-i-2}$. Then $N : V_{i+2} \hookrightarrow V_i$ and $V_i = NV_{i+2} \oplus V(i)$. Thus

$$V = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j V(i),$$

and for $0 \leq j \leq i$ the map $N^j : V(i) \to V_{i-2j}$ is injective. Also note that as a virtual W_K-module $[V(i)] = [V_i] - [V_{i+2} \otimes |\mathrm{Art}_K^{-1}|_K]$. Thus if r is semisimple then (V, r) determines (V, r, N) up to isomorphism. This establishes the fourth part.

Now consider the fifth part. If W is a direct summand it is certainly pure of the same weight k and $\bigwedge^{\dim W} W$ is then pure of weight $k \dim W$. Conversely if W is pure of weight k then

$$W = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j W(i),$$

where $W(i) = W \cap V(i)$. As a W_K-module we can decompose $V(i) = W(i) \oplus U(i)$. Setting

$$U = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j U(i),$$

we see that $V = W \oplus U$ as Weil-Deligne representations. Now suppose only that $\bigwedge^{\dim W} W$ is pure of weight $k \dim W$. Write

$$W \cong \bigoplus_j \mathrm{Sp}_{s_j}(X_j)$$

where each X_j is strictly pure of some weight $k + k_j + (s_j - 1)$. Then, looking at highest exterior powers, we see that $\sum_j k_j = 0$. On the other hand as V is pure we see that $k_j \leq 0$ for all j. We conclude that $k_j = 0$ for all j and hence that W is pure of weight k.

The final part follows from the fifth part by a simple inductive argument. $\square$

Now let L denote a number field. Write $|\ |_L$ for

$$\prod_x |\ |_{L_x} : \mathbb{A}_L^\times / L^\times \longrightarrow \mathbb{R}_{>0}^\times,$$

and write Art_L for

$$\prod_x \mathrm{Art}_{L_x} : \mathbb{A}_L^\times / L^\times \twoheadrightarrow \mathrm{Gal}\,(L^{ac}/L)^{\mathrm{ab}}.$$

We will call a continuous representation

$$R : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

pure of weight k if for all but finitely many finite places x of L the representation R is unramified at x and every eigenvalue α of $R(\mathrm{Frob}_x)$ is a Weil $(\#k(x))^k$-number. Note that if $n = 1$ then R is pure of weight k if and only if for all $\iota : \mathbb{Q}_l^{ac} \hookrightarrow \mathbb{C}$ we have $|\iota R \circ \mathrm{Art}_L|^2 = |\ |_L^{-k}$. In particular if $n = 1$ and R is pure then $R|_{W_{L_x}}$ is strictly pure for all finite places x of L.

We have the following lemma.

Lemma 1.4. *Suppose that M/L is a finite extension of number fields. Suppose also that*

$$R : \mathrm{Gal}\,(L^{ac}/L) \longrightarrow GL_n(\mathbb{Q}_l^{ac})$$

is a continuous semisimple representation which is pure of weight k. Suppose that

$$S : \mathrm{Gal}\,(M^{ac}/M) \longrightarrow GL_{an}(\mathbb{Q}_l^{ac})$$

is another continuous representation with $S^{\mathrm{ss}} \cong R|^a_{\mathrm{Gal}\,(M^{ac}/M)}$ for some $a \in \mathbb{Z}_{>0}$. Suppose finally that w is a place of M above a finite place v of L. If $\mathrm{WD}(S|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})$ is pure of weight k, then $\mathrm{WD}(R|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})$ is also pure of weight k.

Proof: Write

$$R|_{\mathrm{Gal}\,(M^{ac}/M)} = \bigoplus_i R_i$$

where each R_i is irreducible. Then $\det R_i$ is pure of weight $k \dim R_i$ and so that the top exterior power $\bigwedge^{\dim R_i} \mathrm{WD}(R_i|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})$ is also pure of weight $k \dim R_i$. Lemma 1.3(6) tells us that

$$\mathrm{WD}(S|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}} \cong \Big(\bigoplus_i \mathrm{WD}(R_i|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}}\Big)^a \cong \big(\mathrm{WD}(R|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}}\big)^a,$$

and that $\mathrm{WD}(R|_{\mathrm{Gal}\,(M_w^{ac}/M_w)})^{F\text{-ss}}$ is pure of weight w. Applying lemma 1.3(1) and (2), we see that $\mathrm{WD}(R|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})$ is also pure. □

2. Shimura varieties

In this section we study the geometry of integral models of Shimura varieties of the type considered in [HT], but with Iwahori level. It may be viewed as a generalisation of the work of Deligne-Rapoport [DR] in the case of modular curves.

In this section,

- let E be an imaginary quadratic field, F^+ a totally real field and set $F = EF^+$;
- let p be a rational prime which splits $p = uu^c$ in E;
- and let $w = w_1, ..., w_r$ be the primes of F above u;
- and let B be a division algebra with centre F such that
  - $\dim_F B = n^2$,
  - $B^{\mathrm{op}} \cong B \otimes_{F,c} F$,
  - at every place x of F either B_x is split or a division algebra,
  - if n is even then the number of finite places of F^+ above which B is ramified is congruent to $1 + \frac{n}{2}[F^+ : \mathbb{Q}]$ modulo 2.

Pick a positive involution $*$ on B with $*|_F = c$. Let $V = B$ as a $B \otimes_F B^{\mathrm{op}}$-module. For $\beta \in B^{*=-1}$ define a pairing

$$\begin{array}{rcl} (\ ,\ ) : V \times V & \longrightarrow & \mathbb{Q} \\ (x_1, x_2) & \longmapsto & \mathrm{tr}_{F/\mathbb{Q}} \mathrm{tr}_{B/F}(x_1 \beta x_2^*). \end{array}$$

Also define an involution $\#$ on B by $x^\# = \beta x^* \beta^{-1}$ and a reductive group $G/\mathbb{Q}$ by setting, for any $\mathbb{Q}$-algebra R, the group $G(R)$ equal to the set of

$$(\lambda, g) \in R^\times \times (B^{\mathrm{op}} \otimes_{\mathbb{Q}} R)^\times$$

such that

$$gg^\# = \lambda.$$

Let $\nu : G \to \mathbb{G}_m$ denote the multiplier character sending (λ, g) to λ. Note that if x is a rational prime which splits $x = yy^c$ in E then

$$\begin{array}{rcl} G(\mathbb{Q}_x) & \xrightarrow{\sim} & (B_y^{\mathrm{op}})^\times \times \mathbb{Q}_x^\times \\ (\lambda, g) & \longmapsto & (g_y, \lambda). \end{array}$$

We can and will assume that

- if x is a rational prime which does not split in E the $G \times \mathbb{Q}_x$ is quasisplit;
- the pairing $(\ ,\ )$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ has invariants $(1, n-1)$ at one embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and invariants $(0, n)$ at all other embeddings $F^+ \hookrightarrow \mathbb{R}$.

(See section I.7 of [HT] for details.)

Let U be an open compact subgroup of $G(\mathbb{A}^\infty)$. Define a functor $\mathfrak{X}_U$ from the category of pairs (S, s), where S is a connected locally noetherian F-scheme and s is a geometric point of S, to the category of sets, sending (S, s) to the set of isogeny classes of four-tuples $(A, \lambda, i, \overline{\eta})$ where

- A/S is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n^2$;
- $i : B \hookrightarrow \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathrm{Lie}\, A \otimes_{(E \otimes_{\mathbb{Q}} \mathcal{O}_S), 1 \otimes 1} \mathcal{O}_S$ is locally free over $\mathcal{O}_S$ of rank n and the two actions of F^+ coincide;
- $\lambda : A \to A^\vee$ is a polarisation such that for all $b \in B$ we have $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$;
- $\overline{\eta}$ is a $\pi_1(S, s)$-invariant U-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s$ which take the standard pairing $(\ ,\ )$ on V to a $(\mathbb{A}^\infty)^\times$-multiple of the λ-Weil pairing on VA_s.

Here $VA_s = \left(\lim_{\substack{\longleftarrow \\ N}} A[N](k(s)) \right) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the adelic Tate module. For the precise notion of isogeny class see section III.1 of [HT]. If s and s' are both geometric points of a connected locally noetherian F-scheme S then $\mathfrak{X}_U(S, s)$ and $\mathfrak{X}_U(S, s')$ are in canonical bijection. thus we may think of $\mathfrak{X}_U$ as a functor from connected locally noetherian F-schemes to sets. We may further extend it to a functor from all locally noetherian F-schemes to sets by setting

$$\mathfrak{X}_U\left(\coprod_i S_i\right) = \prod_i \mathfrak{X}_U(S_i).$$

If U is sufficiently small (i.e. for some finite place x of $\mathbb{Q}$ the projection of U to $G(\mathbb{Q}_x)$ contains no element of finite order except 1) then $\mathfrak{X}_U$ is represented by a smooth projective variety X_U/F of dimension $n-1$. The inverse system of the X_U for varying U has a natural action of $G(\mathbb{A}^\infty)$.

Choose a maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ of B with $\mathcal{O}_B^* = \mathcal{O}_B$. Also fix an isomorphism $\mathcal{O}_{B,w}^{\mathrm{op}} \cong M_n(\mathcal{O}_{F,w})$, and let $\varepsilon \in B_w$ denote the element corresponding to the diagonal matrix $(1,0,0,...,0) \in M_n(\mathcal{O}_{F,w})$. We decompose $G(\mathbb{A}^\infty)$ as

$$G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,p}) \times \left(\prod_{i=2}^{r}(B_{w_i}^{\mathrm{op}})^\times\right) \times GL_n(F_w) \times \mathbb{Q}_p^\times. \tag{1}$$

Let ϖ denote a uniformiser for $\mathcal{O}_{F,w}$. For $m=(m_2,...,m_r) \in \mathbb{Z}_{\geq 0}^{r-1}$, set

$$U_p^w(m) = \prod_{i=2}^{r} \ker\big((\mathcal{O}_{B,w_i}^{\mathrm{op}})^\times \to (\mathcal{O}_{B,w_i}^{\mathrm{op}}/w_i^{m_i})^\times\big) \subset \prod_{i=2}^{r}(B_{w_i}^{\mathrm{op}})^\times.$$

Let B_n denote the Borel subgroup of GL_n consisting of upper triangular matrices and let N_n denote its unipotent radical. Let $\mathrm{Iw}_{n,w}$ denote the subgroup of $GL_n(\mathcal{O}_{F,w})$ consisting of matrices which reduce modulo w to $B_n(k(w))$. We will consider the following open subgroups of $G(\mathbb{Q}_p)$:

$$\begin{array}{rcl}\mathrm{Ma}(m) & = & U_p^w(m) \times GL_n(\mathcal{O}_{F,w}) \times \mathbb{Z}_p^\times \\ \mathrm{Iw}(m) & = & U_p^w(m) \times \mathrm{Iw}_{n,w} \times \mathbb{Z}_p^\times\end{array}$$

Let U^p be an open compact subgroup of $G(\mathbb{A}^{\infty,p})$. Write U_0 (resp. U) for $U^p \times \mathrm{Ma}(m)$ (resp. $U^p \times \mathrm{Iw}(m)$).

We recall that in section III.4 [HT] integral model of X_{U_0} over $\mathcal{O}_{F,w}$ is defined. It represents the functor $\mathfrak{X}_{U_0}$ from locally noetherian $\mathcal{O}_{F,w}$-schemes to sets. As above, $\mathfrak{X}_{U_0}$ is initially defined on the category of connected locally noetherian $\mathcal{O}_{F,w}$ schemes with a geometric point to sets. It sends (S,s) to the set of prime-to-p isogeny classes of $(r+3)$-tuples $(A,\lambda,i,\overline{\eta}^p,\alpha_i)$, where

- A/S is an abelian scheme of dimension $[F^+:\mathbb{Q}]n^2$;
- $i:\mathcal{O}_B \hookrightarrow \mathrm{End}\,(A)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ such that $\mathrm{Lie}\,A \otimes_{(\mathcal{O}_{E,u}\otimes_{\mathbb{Z}_p}\mathcal{O}_S),1\otimes 1}\mathcal{O}_S$ is locally free of rank n and the two actions of $\mathcal{O}_F$ coincide;
- $\lambda: A \to A^\vee$ is a prime-to-p polarisation such that for all $b \in \mathcal{O}_B$ we have $\lambda\circ i(b) = i(b^*)^\vee \circ \lambda$;

- $\overline{\eta}^p$ is a $\pi_1(S,s)$-invariant U^p-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s$ which take the standard pairing $(\ ,\ )$ on V to a $(\mathbb{A}^{\infty,p})^\times$-multiple of the λ-Weil pairing on $V^p A_s$;
- for $2 \leq i \leq r$, $\alpha_i : (w_i^{-m_i}\mathcal{O}_{B,w_i}/\mathcal{O}_{B,w_i})_S \xrightarrow{\sim} A[w_i^{m_i}]$ is an isomorphism of S-schemes with $\mathcal{O}_B$-actions;

Then X_{U_0} is smooth and projective over $\mathcal{O}_{F,w}$ ([HT], page 109). As U^p varies, the inverse system of the X_{U_0}'s has an action of $G(\mathbb{A}^{\infty,p})$.

Given an $(r+3)$-tuple as above we will write $\mathcal{G}_A$ for $\varepsilon A[w^\infty]$ a Barsotti-Tate $\mathcal{O}_{F,w}$-module. Over a base in which p is nilpotent it is one dimensional. If $\mathcal{A}$ denotes the universal abelian scheme over X_{U_0}, we will write $\mathcal{G}$ for $\mathcal{G}_{\mathcal{A}}$. This $\mathcal{G}$ is *compatible*, i.e. the two actions of $\mathcal{O}_{F,w}$ on $\operatorname{Lie}\mathcal{G}$ coincide (see [HT]).

Write $\overline{X}_{U_0}$ for the special fibre $X_{U_0} \times_{\operatorname{Spec}\mathcal{O}_{F,w}} \operatorname{Spec} k(w)$. For $0 \leq h \leq n-1$, we let $\overline{X}_{U_0}^{[h]}$ denote the reduced closed subscheme of $\overline{X}_{U_0}$ whose closed geometric points s are those for which the maximal etale quotient of $\mathcal{G}_s$ has $\mathcal{O}_{F,w}$-height at most h, and let

$$\overline{X}_{U_0}^{(h)} = \overline{X}_{U_0}^{[h]} - \overline{X}_{U_0}^{[h-1]}$$

(where we set $\overline{X}_{U_0}^{[-1]} = \emptyset$). Then $\overline{X}_{U_0}^{(h)}$ is smooth of pure dimension h (corollary III.4.4 of [HT]), and on it there is a short exact sequence

$$(0) \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^{\mathrm{et}} \longrightarrow (0)$$

where $\mathcal{G}^0$ is a formal Barsotti-Tate $\mathcal{O}_{F,w}$-module and $\mathcal{G}^{\mathrm{et}}$ is an etale Barsotti-Tate $\mathcal{O}_{F,w}$-module with $\mathcal{O}_{F,w}$-height h.

Lemma 2.1. *If $0 \leq h \leq n-1$ then the Zariski closure of $\overline{X}_{U_0}^{(h)}$ contains $\overline{X}_{U_0}^{(0)}$.*

Proof: This is 'well known', but for lack of a reference we give a proof. Let x be a closed geometric point of $\overline{X}_{U_0}^{(0)}$. By lemma II.4.1 of [HT] the formal completion of $\overline{X}_{U_0} \times \operatorname{Spec} k(w)^{ac}$ at x is isomorphic to the equicharacteristic universal deformation ring of $\mathcal{G}_x$. According to the proof of proposition 4.2 of [Dr] this is $\operatorname{Spf} k(w)^{ac}[[T_1,...,T_{n-1}]]$ and we can choose the T_i and a formal parameter S on the universal deformation of $\mathcal{G}_x$ such that

$$[\varpi_w](S) \equiv \varpi_w S + \sum_{i=1}^{n-1} T_i S^{\#k(w)^i} + S^{\#k(w)^n} \pmod{S^{\#k(w)^n+1}}.$$

Thus we get a morphism

$$\operatorname{Spec} k(w)^{ac}[[T_1,...,T_{n-1}]] \longrightarrow \overline{X}_{U_0}$$

lying over $x : k(w)^{ac} \to \overline{X}_{U_0}$, such that, if k denotes the algebraic closure of the field of fractions of $k(w)^{ac}[[T_1,...,T_{n-1}]]/(T_1,...,T_{n-h-1})$, then the induced map

$$\operatorname{Spec} k \longrightarrow \overline{X}_{U_0}$$

factors through $\overline{X}_{U_0}^{(h)}$. Thus x is in the closure of $\overline{X}_{U_0}^{(h)}$, and the lemma follows. $\Box$

Now we define the functor $\mathfrak{X}_U$. Again we initially define it as a functor from the category of connected locally noetherian schemes with a geometric point to sets, but then (as above) we extend it to a functor from locally noetherian schemes to sets. The functor $\mathfrak{X}_U$ will send (S, s) to the set of prime-to-p isogeny classes of $(r+4)$-tuples $(A, \lambda, i, \overline{\eta}^p, \mathcal{C}, \alpha_i)$, where $(A, \lambda, i, \overline{\eta}^p, \alpha_i)$ is as in the definition of gX_{U_0} and $\mathcal{C}$ is a chain of isogenies

$$\mathcal{C} : \mathcal{G} = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}/\mathcal{G}[w]$$

of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each of degree $\#k(w)$ and with composite equal to the canonical map $\mathcal{G} \to \mathcal{G}/\mathcal{G}[w]$. There is a natural transformation of functors $\mathfrak{X}_U \to \mathfrak{X}_{U_0}$.

Lemma 2.2. *The functor $\mathfrak{X}_U$ is represented by a scheme X_U which is finite over X_{U_0}. The scheme X_U has some irreducible components of dimension n.*

Proof: By denoting the kernel of $\mathcal{G}_0 \to \mathcal{G}_j$ by $\mathcal{K}_j \subset \mathcal{G}[w]$, we can view the above chain as a flag

$$0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = \mathcal{G}[w]$$

of closed finite flat subgroup schemes with $\mathcal{O}_{F,w}$-action, with each $\mathcal{K}_j/\mathcal{K}_{j-1}$ having order $\#k(w)$. Let $\mathcal{H}$ denote the sheaf of Hopf algebras over X_{U_0} defining $\mathcal{G}[w]$. Then $\mathfrak{X}_U$ is represented by a closed subscheme X_U of the Grassmanian of chains of locally free direct summands of $\mathcal{H}$. (The closed conditions require that the subsheaves are sheaves of ideals defining a flag of closed subgroup schemes with the desired properties.) Thus X_U is projective over $\mathcal{O}_{F,w}$. At each closed geometric point s of X_{U_0} the number of possible $\mathcal{O}_{F,w}$-submodules of $\mathcal{G}[w]_s \cong \mathcal{G}[w]_s^0 \times \mathcal{G}[w]_s^{\mathrm{et}}$ is finite, so X_U is finite over X_{U_0}. To see that X_U has some components of dimension n it suffices to note that on the generic fibre the map to X_{U_0} is finite etale. $\Box$

We say an isogeny $\mathcal{G} \to \mathcal{G}'$ of one-dimensional compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules over a scheme S of characterstic p has *connected kernel* if it induces the zero map on $\mathrm{Lie}\, \mathcal{G}$. We will denote the relative Frobenius map by $F : \mathcal{G} \to \mathcal{G}^{(p)}$ and let $f = [k(w) : \mathbb{F}_p]$, and then $F^f : \mathcal{G} \to \mathcal{G}^{(\#k(w))}$ is an isogeny of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules of degree $\#k(w)$ and has connected kernel.

We have the following rigidity lemma.

Lemma 2.3. *Let W denote the ring of integers of the completion of the maximal unramified extension of F_w. Suppose that R is an Artinian local W-algebra with residue field $k(w)^{ac}$. Suppose also that*

$$\mathcal{C} : \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}_0$$

is a chain of isogenies of degree $\#k(w)$ of one-dimensional compatible formal Barsotti-Tate $\mathcal{O}_{F,w}$-modules of $\mathcal{O}_{F,w}$-height n with composite equal to multiplication by ϖ_w. If every isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel (for $i = 1, ..., n$) then R is a $k(w)^{ac}$-algebra and $\mathcal{C}$

is the pull-back of a chain of Barsotti-Tate $\mathcal{O}_{F,w}$-modules over $k(w)^{ac}$, with all the isogenies isomorphic to F^f.

Proof: As the composite of the n isogenies induces multiplication by ϖ_w on the tangent space, $\varpi_w = 0$ in R, i.e. R is a $k(w)^{ac}$-algebra. Choose a parameter T_i for $\mathcal{G}_i$ over R. With respect to these choices, let $f_i(T_i) \in R[[T_i]]$ represent $\mathcal{G}_{i-1} \to \mathcal{G}_i$. We can write $f_i(T_i) = g_i(T_i^{p^{h_i}})$ with $h_i \in \mathbb{Z}_{\geq 0}$ and $g_i'(0) \neq 0$. (See [F], chapter I, §3, Theorem 2.) As $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel, $f_i'(0) = 0$ and $h_i > 0$. As f_i commutes with the action $[r]$ for all $r \in \mathcal{O}_{F,w}$, we have $\overline{r}^{p^{h_i}} = \overline{r}$ for all $\overline{r} \in k(w)$, hence h_i is a multiple of $f = [k(w) : \mathbb{F}_p]$. Reducing modulo the maximal ideal of R we see that $h_i \leq f$ and so in fact $h_i = f$ and $g_i'(0) \in R^\times$. Thus $\mathcal{G}_i \cong \mathcal{G}_0^{(\#k(w)^i)}$ in such a way that the isogeny $\mathcal{G}_0 \to \mathcal{G}_i$ is identified with F^{fi}. In particular $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^n)}$ and hence $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^{nm})}$ for any $m \in \mathbb{Z}_{\geq 0}$. As R is Artinian some power of the absolute Frobenius on R factors through $k(w)^{ac}$. Thus $\mathcal{G}_0$ is a pull-back from $k(w)^{ac}$ and the lemma follows. $\square$

Now let $Y_{U,i}$ denote the closed subscheme of $\overline{X}_U = X_U \times_{\operatorname{Spec} \mathcal{O}_{F,w}} \operatorname{Spec} k(w)$ over which $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel.

Proposition 2.4. (1) *X_U has pure dimension n and semistable reduction over $\mathcal{O}_{F,w}$, that is, for all closed points x of the special fibre $X_U \times_{\operatorname{Spec} \mathcal{O}_{F,w}} \operatorname{Spec} k(w)$, there exists an etale morphism $V \to X_U$ with $x \in \operatorname{Im} V$ and an etale $\mathcal{O}_{F,w}$-morphism:*

$$V \longrightarrow \operatorname{Spec} \mathcal{O}_{F,w}[T_1, ..., T_n]/(T_1 \cdots T_m - \varpi_w)$$

for some $1 \leq m \leq n$, where ϖ_w is a uniformizer of $\mathcal{O}_{F,w}$.

(2) *X_U is regular and the natural map $X_U \to X_{U_0}$ is finite and flat.*

(3) *Each $Y_{U,i}$ is smooth over $\operatorname{Spec} k(w)$ of pure dimension $n-1$, $\overline{X}_U = \bigcup_{i=1}^n Y_{U,i}$ and, for $i \neq j$ the schemes $Y_{U,i}$ and $Y_{U,j}$ share no common connected component. In particular, X_U has strictly semistable reduction.*

Proof: In this proof we will make repeated use of the following version of Deligne's homogeneity principle ([DR]). Write W for the ring of integers of the completion of the maximal unramified extension of F_w. In what follows, if s is a closed geometric point of an $\mathcal{O}_{F,w}$-scheme X locally of finite type, then we write $\mathcal{O}_{X,s}^\wedge$ for the completion of the strict Henselisation of X at s, i.e. $\mathcal{O}_{X \times \operatorname{Spec} W, s}^\wedge$. Let $\mathbf{P}$ be a property of complete noetherian local W-algebras such that if X is a $\mathcal{O}_{F,w}$-scheme locally of finite type then the set of closed geometric points s of X for which $\mathcal{O}_{X,s}^\wedge$ has property $\mathbf{P}$ is Zariski open. Also let $X \to X_{U_0}$ be a finite morphism with the following properties

(i) If s is a closed geometric point of $\overline{X}_{U_0}^{(h)}$ then, up to isomorphism, $\mathcal{O}_{X,s}^\wedge$ does not depend on s (but only on h).

(ii) There is a unique geometric point of X above any geometric point of $\overline{X}_{U_0}^{(0)}$.

If $\mathcal{O}^{\wedge}_{X,s}$ has property $\mathbf{P}$ for every geometric point of X over $\overline{X}^{(0)}_{U_0}$, then $\mathcal{O}^{\wedge}_{X,s}$ has property $\mathbf{P}$ for every closed geometric point of X. Indeed, if we let Z denote the closed subset of X where $\mathbf{P}$ does not hold, then its image in X_{U_0} is closed and is either empty or contains some $\overline{X}^{(h)}_{U_0}$. In the latter case, by lemma 2.1, it also contains $\overline{X}^{(0)}_{U_0}$, which is impossible. Thus Z must be empty.

Note that both $X = X_U$ and $X = Y_{U,i}$ satisfy the above condition (ii) for the homogeneity principle, by letting $R = k(w)^{ac}$ in lemma 2.3.

(1): The dimension of $\mathcal{O}^{\wedge}_{X_U,s}$ as s runs over geometric points of X_U above $\overline{X}^{(0)}_{U_0}$ is constant, say m. Applying the homogeneity principle to $X = X_U$ with $\mathbf{P}$ being 'dimension m', we see that X_U has pure dimension m. By lemma 2.2 we must have $m = n$ and X_U has pure dimension n.

Now we will apply the above homogeneity principle to $X = X_U$ taking $\mathbf{P}$ to be 'isomorphic to $W[[T_1, ..., T_n]]/(T_1 \cdots T_m - \varpi_w)$ for some $m \leq n$'. By a standard argument (see e.g. the proof of proposition 4.10 of [Y]) the set of points with this property is open and if all closed geometric points of X_U have this property $\mathbf{P}$ then X_U is semistable of pure dimension n.

Let s be a geometric point of X_U over a point of $\overline{X}^{(0)}_{U_0}$. Choose a basis e_i of $\mathrm{Lie}\,\mathcal{G}_i$ over $\mathcal{O}^{\wedge}_{X_U,s}$ such that e_n maps to e_0 under the isomorphism $\mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[w] \xrightarrow{\sim} \mathcal{G}_0$ induced by ϖ_w. With respect to these bases let $X_i \in \mathcal{O}^{\wedge}_{X_U,s}$ represent the linear map $\mathrm{Lie}\,\mathcal{G}_{i-1} \to \mathrm{Lie}\,\mathcal{G}_i$. Then

$$X_1 \cdots X_n = \varpi_w.$$

Moreover it follows from lemma 2.3 that $\mathcal{O}^{\wedge}_{X_U,s}/(X_1, ..., X_n) = k(w)^{ac}$. (Because, by lemma III.4.1 of [HT], $\mathcal{O}^{\wedge}_{X_{U_0},s}$ is the universal deformation space of $\mathcal{G}_s$. Hence by lemma 2.3, $\mathcal{O}^{\wedge}_{X_U,s}$ is the universal deformation space for the chain

$$\mathcal{G}_s \xrightarrow{F^f} \mathcal{G}_s^{(\#k(w))} \xrightarrow{F^f} \cdots \xrightarrow{F^f} \mathcal{G}_s^{(\#k(w)^n)} \cong \mathcal{G}_s/\mathcal{G}_s[\varpi_w].)$$

Thus we get a surjection

$$W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w) \twoheadrightarrow \mathcal{O}^{\wedge}_{X_U,s}$$

and as $\mathcal{O}^{\wedge}_{X_U,s}$ has dimension n this map must be an isomorphism.

(2): We see at once that X_U is regular. Then [AK] V, 3.6 tells us that $X_U \to X_{U_0}$ is flat.

(3): We apply the homogeneity principle to $X = Y_{U,i}$ taking $\mathbf{P}$ to be 'formally smooth of dimension $n-1$'. If s is a geometric point of $Y_{U,i}$ above $\overline{X}^{(0)}_{U_0}$ then we see that $\mathcal{O}^{\wedge}_{Y_{U,i},s}$ is cut out in $\mathcal{O}^{\wedge}_{X_U,s} \cong W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w)$ by the single equation $X_i = 0$. (We are using the parameters X_i defined above.) Thus

$$\mathcal{O}^{\wedge}_{Y_{U,i},s} \cong k(w)^{ac}[[X_1, ..., X_{i-1}, X_{i+1}, ..., X_n]]$$

is formally smooth of dimension $n-1$. We deduce that $Y_{U,i}$ is smooth of pure dimension $n-1$.

As our $\mathcal{G}/\overline{X}_U$ is one-dimensional, over a closed point, at least one of the isogenies $\mathcal{G}_{i-1} \to \mathcal{G}_i$ must have connected kernel, which shows that $\overline{X}_U = \bigcup_i Y_{U,i}$. Suppose $Y_{U,i}$ and $Y_{U,j}$ share a connected componet Y for some $i \neq j$. Then Y would be finite flat over $\overline{X}_{U_0}$ and so the image of Y would meet $\overline{X}_{U_0}^{(n-1)}$. This is impossible as above a closed point of $\overline{X}_{U_0}^{(n-1)}$ one isogeny among the chain can have connected kernel. Thus, for $i \neq j$ the closed subschemes $Y_{U,i}$ and $Y_{U,j}$ have no connected component in common. □

By the strict semistability, if we write, for $S \subset \{1, ..., n\}$,

$$Y_{U,S} = \bigcap_{i \in S} Y_{U,i}, \quad Y_{U,S}^0 = Y_{U,S} - \bigcup_{T \supsetneq S} Y_{U,T}$$

then $Y_{U,S}$ is smooth over $\operatorname{Spec} k(w)$ of pure dimension $n - \#S$ and $Y_{U,S}^0$ are disjoint for different S. With respect to the finite flat map $\overline{X}_U \to \overline{X}_{U_0}$, the inverse image of $\overline{X}_{U_0}^{[h]}$ is exactly the locus where at least $n-h$ of the isogenies have connected kernel, i.e. $\bigcup_{\#S \geq n-h} Y_{U,S}$. Hence the inverse image of $\overline{X}_{U_0}^{(h)}$ is equal to $\bigcup_{\#S=n-h} Y_{U,S}^0$. Also note that the inverse system of $Y_{U,S}^0$ for varying U^p is stable by the action of $G(\mathbb{A}^{\infty,p})$.

Next we will relate the open strata $Y_{U,S}^0$ to the Igusa varieties of the first kind defined in [HT]. For $0 \leq h \leq n-1$ and $m \in \mathbb{Z}_{\geq 0}^r$, we write $I_{U^p,m}^{(h)}$ for the Igusa varities of the first kind defined on page 121 of [HT]. We also define an *Iwahori-Igusa variety of the first kind*

$$I_U^{(h)}/\overline{X}_{U_0}^{(h)}$$

as the moduli space of chains of isogenies

$$\mathcal{G}^{\mathrm{et}} = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_h = \mathcal{G}^{\mathrm{et}}/\mathcal{G}^{\mathrm{et}}[w]$$

of etale Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each isogeny having degree $\#k(w)$ and with composite equal to the natural map $\mathcal{G}^{\mathrm{et}} \to \mathcal{G}^{\mathrm{et}}/\mathcal{G}^{\mathrm{et}}[w]$. Then $I_U^{(h)}$ is finite etale over $\overline{X}_{U_0}^{(h)}$, and as the Igusa variety $I_{U_p,(1,m)}^{(h)}$ classifies the isomorphisms

$$\alpha_1^{\mathrm{et}} : (w^{-1}\mathcal{O}_{F,w}/\mathcal{O}_{F,w})^h_{\overline{X}_{U_0}^{(h)}} \longrightarrow \mathcal{G}^{\mathrm{et}}[w],$$

the natural map

$$I_{U^p,(1,m)}^{(h)} \longrightarrow I_U^{(h)}$$

is finite etale and Galois with Galois group $B_h(k(w))$. Hence we can identify $I_U^{(h)}$ with $I_{U^p,(1,m)}^{(h)}/B_h(k(w))$. Note that the system $I_U^{(h)}$ for varying U^p naturally inherits the action of $G(\mathbb{A}^{\infty,p})$.

Lemma 2.5. *For $S \subset \{1, ..., n\}$ with $\#S = n-h$, there exists a finite map of $\overline{X}_{U_0}^{(h)}$-schemes*

$$\varphi : Y_{U,S}^0 \longrightarrow I_U^{(h)}$$

which is bijective on the geometric points.

Proof: The map is defined in a natural way from the chain of isogenies $\mathcal{C}$ by passing to the etale quotient $\mathcal{G}^{\mathrm{et}}$, and it is finite as $Y^0_{U,S}$ (resp. $I^{(h)}_U$) is finite (resp. finite etale) over $\overline{X}^{(h)}_{U_0}$. Let s be a closed geometric point of $I^{(h)}_U$ with a chain of isogenies

$$\mathcal{G}^{\mathrm{et}}_s = \mathcal{G}^{\mathrm{et}}_0 \to \cdots \to \mathcal{G}^{\mathrm{et}}_h = \mathcal{G}^{\mathrm{et}}_s/\mathcal{G}^{\mathrm{et}}_s[w].$$

For $1 \leq i \leq n$ let $j(i)$ denote the number of elements of S which are less than or equal to i. Set $\mathcal{G}_i = (\mathcal{G}^0_s)^{(\# k(w)^{j(i)})} \times \mathcal{G}^{\mathrm{et}}_{i-j(i)}$. If $i \notin S$, define an isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ to be the identity times the given isogeny $\mathcal{G}^{\mathrm{et}}_{i-1} \to \mathcal{G}^{\mathrm{et}}_i$. If $i \in S$, define an isogeny $\mathcal{G}_{i-1} \to \mathcal{G}_i$ to be F^f times the identity. Then

$$\mathcal{G}_0 \to \cdots \to \mathcal{G}_n$$

defines the unique geometric point of $Y^0_{U,S}$ above s. $\square$

Now recall from [HT] III.2, that for an irreducible algebraic representation ξ of G over $\mathbb{Q}^{ac}_l$, one can associate a lisse $\mathbb{Q}^{ac}_l$-sheaf $\mathcal{L}_\xi/X_U$ for every U such that X_U is defined, and the action of $G(\mathbb{A}^{\infty,p})$ extends to $\mathcal{L}_\xi$. The sheaf $\mathcal{L}_\xi$ is extended to the integral models and Igusa varieties, and on $I^{(h)}_{U^p,(1,m)}$, $I^{(h)}_U$ and $Y^0_{U,S}$ they are the pull back of $\mathcal{L}_\xi$ on $\overline{X}^{(h)}_{U_0}$.

Corollary 2.6. *For every $i \in \mathbb{Z}_{\geq 0}$, we have isomorphisms*

$$\begin{aligned} H^i_c(Y^0_{U,S} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) &\xrightarrow{\sim} H^i_c(I^{(h)}_U \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) \\ &\xrightarrow{\sim} H^i_c(I^{(h)}_{U^p,(1,m)} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi)^{B_h(k(w))} \end{aligned}$$

that are compatible with the actions of $G(\mathbb{A}^{\infty,p})$ when we vary U^p.

Proof: By lemma 2.5, for any lisse $\mathbb{Q}^{ac}_l$-sheaf $\mathcal{F}$ on $I^{(h)}_U$, we have $\mathcal{F} \cong \varphi_*\varphi^*\mathcal{F}$ by looking at the stalks at all geometric points. As φ is finite the first isomorphism follows. The second isomorphism follows easilly as $I^{(h)}_{U^p,(1,m)} \to I^{(h)}_U$ is finite etale and Galois with Galois group $B_h(k(w))$. $\square$

In the next section, we will be interested in the $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}^{\mathbb{Z}}_w$-modules

$$H^i(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi) = \varinjlim_{U^p} H^i(Y_{U,S} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi).$$

Here we relate the alternating sum of these modules to the cohomology of Igusa varieties. We will define the elements of Groth$\,(G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}^{\mathbb{Z}}_w)$ (we write Groth$\,(G)$ for the

Grothendieck group of admissible G-modules) as follows:

$$\begin{aligned}\left[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)\right] &= \sum_i (-1)^{n-\#S-i} H^i(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi),\\ \left[H_c(Y^0_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)\right] &= \sum_i (-1)^{n-\#S-i} \varinjlim_{U^p} H^i_c(Y^0_{U,S}\times_{k(w)} k(w)^{ac},\mathcal{L}_\xi),\\ \left[H_c(I^{(h)}_{\mathrm{Iw}(m)},\mathcal{L}_\xi)\right] &= \sum_i (-1)^{h-i} \varinjlim_{U^p} H^i_c(I^{(h)}_U\times_{k(w)} k(w)^{ac},\mathcal{L}_\xi).\end{aligned}$$

Then, because

$$Y_{U,S} = \bigcup_{T\supset S} Y^0_{U,T}$$

for each $U = U^p \times \mathrm{Iw}(m)$, we have equalities

$$\begin{aligned}\left[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)\right] &= \sum_{T\supset S} (-1)^{(n-\#S)-(n-\#T)} \left[H_c(Y^0_{\mathrm{Iw}(m),T},\mathcal{L}_\xi)\right]\\ &= \sum_{T\supset S} (-1)^{(n-\#S)-(n-\#T)} \left[H_c(I^{(n-\#T)}_{\mathrm{Iw}(m)},\mathcal{L}_\xi)\right].\end{aligned}$$

As there are $\binom{n-\#S}{h}$ subsets T with $\#T = n-h$ and $T\supset S$, we conclude:

Lemma 2.7. *We have an equality*

$$\left[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)\right] = \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h}\binom{n-\#S}{h}\left[H_c(I^{(h)}_{\mathrm{Iw}(m)},\mathcal{L}_\xi)\right]$$

in the Grothendieck group of admissible $G(\mathbb{A}^{\infty,p})\times \mathrm{Frob}_w^{\mathbb{Z}}$-modules over $\mathbb{Q}_l^{ac}$.

3. Proof of the main theorem

We now return to the situation in theorem 1.1. Recall that L is an imaginary CM field and that Π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi\circ c \cong \Pi^\vee$;
- Π_∞ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from L to $\mathbb{Q}$ of GL_n;
- and for some finite place x of L the representation Π_x is square integrable.

Recall also that v is a place of L above a rational prime p, that $l\neq p$ is a second rational prime and that $\imath : \mathbb{Q}_l^{ac} \xrightarrow{\sim} \mathbb{C}$. Recall finally that $R_l(\Pi)$ is the l-adic representation associated to Π.

Choose a quadratic CM extension L'/L in which v and x split. Choose places $v'\neq x'$ of L' above v and x respectively. Also choose an imaginary quadratic field E and a totally real field F^+ such that

- $[F^+ : \mathbb{Q}]$ is even;
- $F = EF^+$ is soluble and Galois over L';
- p splits as uu^c in E;
- there is a place w of F above u and v' such that $\Pi_{F,w}$ has an Iwahori fixed vector;
- x lies above a rational prime which splits in E and x' splits in F.

Denote by Π_F the base change of Π to $GL_n(\mathbb{A}_F)$. Note that the component of Π_F at a place above x' is square integrable and hence Π_F is cuspidal.

Choose a division algebra B with centre F as in the previous section and satisfying

- B_x is split for all places $x \neq z, z^c$ of F.

Also choose , β and G as in the previous section. Then it follows from theorem VI.2.9 and lemma VI.2.10 of [HT] that we can find

- a character $\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$,
- an irreducible algebraic representation ξ of G over $\mathbb{Q}_l^{ac}$,
- and an automorphic representation π of $G(\mathbb{A})$,

such that

- π_∞ is cohomological for $\imath\xi$,
- ψ is unramified above p,
- $\psi^c|_{E_\infty^\times}$ is the inverse of the restriction of $\imath\xi$ to $E_\infty^\times \subset G(\mathbb{R})$,
- ψ^c/ψ is the restriction of the central character of Π_F to $\mathbb{A}_E^\times$,
- and if x is a rational prime which splits yy^c in E then $\pi_x = \left(\bigotimes_{z|y} \mathrm{JL}^{-1}(\Pi_z)\right) \otimes \psi_y$ as a representation of $(B_y^{\mathrm{op}})^\times \times \mathbb{Q}_x^\times \cong \left(\bigotimes_{z|y} (B_z^{\mathrm{op}})^\times\right) \times \mathbb{Q}_x^\times$.

Here JL denotes the identity if B_z is split and denotes the Jacquet-Langlands correspondence if B_z is a division algebra. (See section I.3 of [HT].)

We will call two irreducible admissible representations π' and π'' of $G(\mathbb{A}^\infty)$ *nearly equivalent* if $\pi'_x \cong \pi''_x$ for all but finitely many rational primes x. If M is an admissible $G(\mathbb{A}^\infty)$-module and π' is an irreducible admissible representation of $G(\mathbb{A}^\infty)$ then we define the π'-*near isotypic* component $M[\pi']$ of M to be the largest $G(\mathbb{A}^\infty)$-submodule of M all whose irreducible subquotients are nearly equivalent to π'. Then

$$M = \bigoplus M[\pi']$$

as π' runs over near equivalence classes of irreducible admissible $G(\mathbb{A}^\infty)$-modules. (This follows from the following fact. Suppose that A is a (commutative) polynomial algebra over $\mathbb{C}$ in countably many variables, and that M is an A-module which is finitely generated over

$\mathbb{C}$. Then we can write
$$M = \bigoplus_{\mathfrak{m}} M_{\mathfrak{m}},$$
where $\mathfrak{m}$ runs over maximal ideals of A with residue field $\mathbb{C}$.)

We consider the Shimura varieties X_U/F for open compact subgroups U of $G(\mathbb{A}^\infty)$ as in the last section. Then
$$H^i(X, \mathcal{L}_\xi) = \varinjlim_U H^i(X_U \times_F F^{ac}, \mathcal{L}_\xi)$$
is a semisimple, admissible $G(\mathbb{A}^\infty)$-module with a commuting continuous action of the Galois group $\mathrm{Gal}\,(F^{ac}/F)$. (For details see III.2 of [HT].)

The following lemma follows from [HT], particularly corollary VI.2.3, corollary VI.2.7 and theorem VII.1.7.

Lemma 3.1. *Keep the notation and assumptions above. (In particular we are assuming that π arises from a cuspidal automorphic representation Π of $GL_n(\mathbb{A}_F)$.)*

1. *If $i \neq n-1$ then $H^i(X, \mathcal{L}_\xi)[\pi] = (0)$.*
2. *As $G(\mathbb{A}^\infty) \times \mathrm{Gal}\,(F^{ac}/F)$-modules,*
$$H^{n-1}(X, \mathcal{L}_\xi)[\pi] = \bigoplus_{\pi'} \pi' \otimes R'_l(\Pi)^{m(\pi')} \otimes R_l(\psi),$$
where π' runs over irreducible admissible representations of $G(\mathbb{A}^\infty)$ nearly equivalent to π and where $m(\pi') \in \mathbb{Z}_{\geq 0}$, and $R_l(\Pi) = R'_l(\Pi)^{\mathrm{ss}}$.
3. *$m(\pi) > 0$.*
4. *If $m(\pi') > 0$ then $\pi'_p \cong \pi_p$.*

If π' is an irreducible admissible representation of $G(\mathbb{A}^\infty)$ we can decompose it as $(\pi')^p \otimes (\prod_{i=2}^r \pi'_{w_i}) \otimes \pi'_w \otimes \pi'_{p,0}$, corresponding to the decomposition (1). If π'' is an irreducible admissible representation of $G(\mathbb{A}^{\infty,p})$ and N is an admissible $G(\mathbb{A}^{\infty,p})$-module then we can define the π''-near isotypic component of N in the same manner as we did for $G(\mathbb{A}^\infty)$-modules. If M is an admissible $G(\mathbb{A}^\infty)$-module and π' is an irreducible admissible representation of $G(\mathbb{A}^\infty)$ then
$$M^{\mathrm{Iw}(m)}[(\pi')^p] = M[\pi']^{\mathrm{Iw}(m)}.$$
We will write
$$H^i(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi) = \varinjlim_{U^p} H^i(X_{U^p \times \mathrm{Iw}(m)} \times_F F^{ac}, \mathcal{L}_\xi) \cong H^i(X, \mathcal{L}_\xi)^{\mathrm{Iw}(m)}.$$
It is a semisimple admissible $G(\mathbb{A}^{\infty,p})$-module with a commuting continuous action of $\mathrm{Gal}\,(F^{ac}/F)$.

Theorem 3.2. *Keep the above notation and assumptions. (In particular we are assuming that π arises from a cuspidal automorphic representation Π of $GL_n(\mathbb{A}_F)$.) Let U^p be a* *sufficiently small open compact subgroup of* $G(\mathbb{A}^{\infty,p})$*. Then*

$$\mathrm{WD}\big(H^{n-1}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)[\pi^p]^{U^p}\big)$$

is pure.

Proof: As $X_U = X_{U^p\times \mathrm{Iw}(m)}$ is strictly semistable by proposition 2.4, we can use the Rapoport-Zink weight spectral sequence [RZ] to compute $H^{n-1}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)$. For X_U, it reads

$$E_1^{i,j}(U) = \bigoplus_{t\geq \max(0,-i)}\ \bigoplus_{\#S=i+2t+1} H^{j-2t}(Y_{U,S}\times_{k(w)} k(w)^{ac},\mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_U\times_F F_w^{ac},\mathcal{L}_\xi).$$

Passing to the limit with respect to U^p, it gives rise to the following spectral sequence of admissible $G(\mathbb{A}^{\infty,p})\times \mathrm{Frob}_w^{\mathbb{Z}}$-modules

$$E_1^{i,j}(\mathrm{Iw}(m)) = \bigoplus_{t\geq \max(0,-i)}\ \bigoplus_{\#S=i+2t+1} H^{j-2t}(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi).$$

Hence we get a spectral sequence of $\mathrm{Frob}_w^{\mathbb{Z}}$-modules

$$E_1^{i,j}(\mathrm{Iw}(m))[\pi^p]^{U^p} \Rightarrow H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)[\pi^p]^{U^p}. \tag{2}$$

The sheaf $\mathcal{L}_\xi$ is pure, say of weight w_ξ. Thus the action of Frob_w on $E_1^{i,j}$ is pure of weight $w_\xi + j$ by the Weil conjectures. The theory of weight spectral sequence ([RZ]) defines an operator

$$N : E_1^{i,j}(\mathrm{Iw}(m))[\pi^p]^{U^p}(1) \to E_1^{i+2,j-2}(\mathrm{Iw}(m))[\pi^p]^{U^p},$$

which induces the N for $\mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)[\pi^p]^{U^p})$ and has the property that

$$N^i : E_1^{-i,j+i}(\mathrm{Iw}(m))[\pi^p]^{U^p}(i) \xrightarrow{\sim} E_1^{i,j-i}(\mathrm{Iw}(m))[\pi^p]^{U^p}$$

for all i. If the spectral sequence (2) degenerates at E_1, then it follows that the Weil-Deligne representation $\mathrm{WD}\big(H^{n-1}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)[\pi^p]^{U^p}\big)$ is pure of weight $w_\xi+(n-1)$. Thus it suffices to show that

$$E_1^{i,j}(\mathrm{Iw}(m))[\pi^p]^{U^p} = (0)$$

if $i+j\neq n-1$, i.e. that

$$H^j(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)[\pi^p]^{U^p} = (0)$$

if $j \neq n-\#S$.

We first recall some notation from [HT]. For $h=0,...,n-1$ let P_h denote the maximal parabolic in GL_n consisting of matrices $g\in GL_n$ with $g_{ij}=0$ for $i>n-h$ and $j\leq n-h$. Also let N_h denote the unipotent radical of P_h, let P_h^{op} denote the opposite parabolic and let N_h^{op} denote the unipotent radical of P_h^{op}. Let $D_{F_w,n-h}$ denote the division algebra with centre F_w and Hasse invariant $1/(n-h)$. If π' is a square integrable representation of $GL_{n-h}(F_w)$, let $\varphi_{\pi'}$ denote a pseudo-coefficient for π' as in section I.3 of [HT]. (Note that this depends on the choice of a Haar measure, but in the formulae below this choice will always be cancelled by the choice of an associated Haar measure on $D_{F_w,n-h}^\times$. See [HT] for details.)

If we introduce the limit of cohomology groups of Igusa varieties for varying level structure at p as in (see p.136 of [HT]);

$$\big[H_c(I^{(h)},\mathcal{L}_\xi)\big]=\sum_i(-1)^{h-i}\varinjlim_{U^p,m}H_c^i(I^{(h)}_{U^p,m}\times_{k(w)}k(w)^{ac},\mathcal{L}_\xi),$$

then the second isomorphism of corollary 2.6 and theorem V.5.4 of [HT] tell us that

$$\begin{aligned} n\big[H_c(I^{(h)}_{\mathrm{Iw}(m)},\mathcal{L}_\xi)\big] &= n\big[H_c(I^{(h)},\mathcal{L}_\xi)\big]^{U_p^w(m)\times\mathrm{Iw}_{h,w}}\\ &=\sum_i(-1)^{n-1-i}\mathrm{Red}^{(h)}\big[H^i(X,\mathcal{L}_\xi)^{U_p^w(m)}\big]\end{aligned}$$

in $\mathrm{Groth}\,(G(\mathbb{A}^{\infty,p})\times\mathrm{Frob}_w^{\mathbb{Z}})$, where

$$\mathrm{Red}^{(h)}:\mathrm{Groth}\,(GL_n(F_w)\times\mathbb{Q}_p^\times)\longrightarrow\mathrm{Groth}\,(\mathrm{Frob}_w^{\mathbb{Z}})$$

is the composite of the normalised Jacquet functor

$$J_{N_h^{\mathrm{op}}}:\mathrm{Groth}\,(GL_n(F_w)\times\mathbb{Q}_p^\times)\longrightarrow\mathrm{Groth}\,(GL_{n-h}(F_w)\times GL_h(F_w)\times\mathbb{Q}_p^\times)$$

with the functor

$$\mathrm{Groth}\,(GL_{n-h}(F_w)\times GL_h(F_w)\times\mathbb{Q}_p^\times)\longrightarrow\mathrm{Groth}\,(\mathrm{Frob}_w^{\mathbb{Z}})$$

which sends $[\alpha\otimes\beta\otimes\gamma]$ to

$$\sum_\phi\mathrm{vol}\,(D^\times_{F_w,n-h}/F_w^\times)^{-1}\mathrm{tr}\,\alpha(\varphi_{\mathrm{Sp}_{n-h}(\phi)})(\dim\beta^{\mathrm{Iw}_{h,w}})\Big[\mathrm{rec}\big(\phi^{-1}|\ \ |_w^{\frac{1-n}{2}}(\gamma^{\mathbb{Z}_p^\times}\circ\mathbf{N}_{F_w/E_u})^{-1}\big)\Big],$$

where the sum is over characters ϕ of $F_w^\times/\mathcal{O}_{F,w}^\times$. (We just took the $\mathrm{Iw}_{h,w}$-invariant part of the $\mathrm{Red}_1^{(h)}$, which is defined on p.182 of [HT]. Note that Frob_w acts on $H_c(I^{(h)},\mathcal{L}_\xi)$ as

$$(1,p^{-[k(w):\mathbb{F}_p]},-1,1,1)\in G(\mathbb{A}^{\infty,p})\times(\mathbb{Q}_p^\times/\mathbb{Z}_p^\times)\times\mathbb{Z}\times GL_h(F_w)\times\left(\prod_{i=2}^r(B_{w_i}^{\mathrm{op}})^\times\right),$$

where we have identified $D^\times_{F_w,n-h}/\mathcal{O}^\times_{D_{F_w,n-h}}$ with $\mathbb{Z}$ via $w(\det)$.)

In particular, by lemma 3.1(1), we have an equality in $\mathrm{Groth}\,(\mathrm{Frob}_w^{\mathbb{Z}})$:

$$n\big[H_c(I^{(h)}_{\mathrm{Iw}(m)},\mathcal{L}_\xi)[\pi^p]^{U^p}\big]=\mathrm{Red}^{(h)}\big[H^{n-1}(X,\mathcal{L}_\xi)^{U_p^w(m)}[\pi^p]^{U^p}\big].$$

Moreover $H^{n-1}(X,\mathcal{L}_\xi)^{U_p^w(m)}[\pi^p]^{U^p}$ is $\pi_w\otimes\pi_{p,0}$-isotypic as a $GL_n(F_w)\times\mathbb{Q}_p^\times$-module by lemma 3.1(4). As $\pi_w=\Pi_{F,w}$ has an Iwahori fixed vector and $\pi_{p,0}=\psi_u$ is unramified,

$$(\dim\Pi_{F,w}^{\mathrm{Iw}_{n,w}})\big[H^{n-1}(X,\mathcal{L}_\xi)^{U_p^w(m)}[\pi^p]^{U^p}\big]=(\dim H^{n-1}(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p})\big[\Pi_{F,w}\otimes\psi_u\big],$$

and

$$n(\dim\Pi_{F,w}^{\mathrm{Iw}_{n,w}})\big[H_c(I^{(h)}_{\mathrm{Iw}(m)},\mathcal{L}_\xi)[\pi^p]^{U^p}\big]=(\dim H^{n-1}(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p})\,\mathrm{Red}^{(h)}\big[\Pi_{F,w}\otimes\psi_u\big].$$

Combining this with lemma 2.7, we get

$$\begin{aligned} & n(\dim \Pi_{F,w}^{\mathrm{Iw}_{n,w}})\big[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)[\pi^p]^{U^p}\big] \\ &= (\dim H^{n-1}(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p}) \sum_{h=0}^{n-\#S}(-1)^{n-\#S-h}\binom{n-\#S}{h}\mathrm{Red}^{(h)}\big[\Pi_{F,w}\otimes\psi_u\big]. \end{aligned}$$

As $\Pi_{F,w}$ is tempered, it is a full normalised induction of the form

$$\text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\mathrm{Sp}_{s_1}(\pi_1)\otimes\cdots\otimes\mathrm{Sp}_{s_t}(\pi_t)),$$

where π_i is an irreducible cuspidal representation of $GL_{g_i}(F_w)$ and P is a parabolic subgroup of GL_n with Levi component $GL_{s_1g_1}\times\cdots\times GL_{s_tg_t}$. As $\Pi_{F,w}$ has an Iwahori fixed vector, we must have $g_i=1$ and π_i unramified for all i. Note that, for this type of representation (full induced from square integrables $\mathrm{Sp}_{s_i}(\pi_i)$ with π_i an unramified character of $F_w^\times$),

$$\begin{aligned} &\dim\big(\text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\mathrm{Sp}_{s_1}(\pi_1)\otimes\cdots\otimes\mathrm{Sp}_{s_t}(\pi_t))\big)^{\mathrm{Iw}_{n,w}} \\ &= \#P(k(w))\backslash GL_n(k(w))/B_n(k(w)) = \frac{n!}{\prod_j s_j!}. \end{aligned}$$

We can compute $\mathrm{Red}^{(h)}\big[\Pi_{F,w}\otimes\psi_u\big]$ using lemma I.3.9 of [HT] (but note the typo there — "positive integers $h_1,...,h_t$" should read "non-negative integers $h_1,...,h_t$"). Putting $V_i=\mathrm{rec}\big(\pi_i^{-1}|\ \ |_w^{\frac{1-n}{2}}(\psi_u\circ\mathbf{N}_{F_w/E_u})^{-1}\big)$, we see that

$$\begin{aligned} \mathrm{Red}^{(h)}\big[\Pi_{F,w}\otimes\psi_u\big] &= \sum_i \dim\big(\text{n-Ind}_{P'(F_w)}^{GL_h(F_w)}(\mathrm{Sp}_{s_i+h-n}(\pi_i|\ \ |^{n-h})\otimes\bigotimes_{j\neq i}\mathrm{Sp}_{s_j}(\pi_j))\big)^{\mathrm{Iw}_{h,w}}\big[V_i\big] \\ &= \sum_i \frac{h!}{(s_i+h-n)!\prod_{j\neq i}s_j!}\big[V_i\big] \end{aligned}$$

where the sum runs only over those i for which $s_i\geq n-h$, and $P'\subset GL_h$ is a parabolic subgroup. Thus

$$\begin{aligned} & n\frac{n!}{\prod_j s_j!}\big[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)[\pi^p]^{U^p}\big] \\ &= D\sum_{h=0}^{n-\#S}(-1)^{n-\#S-h}\binom{n-\#S}{h}\sum_{i:\,s_i\geq n-h}\frac{h!}{(s_i+h-n)!\prod_{j\neq i}s_j!}\big[V_i\big] \\ &= D\sum_{i=1}^{t}\frac{(n-\#S)!}{(s_i-\#S)!\prod_{j\neq i}s_j!}\sum_{h=n-s_i}^{n-\#S}(-1)^{n-\#S-h}\binom{s_i-\#S}{h+s_i-n}\big[V_i\big] \\ &= D\sum_{s_i=\#S}\frac{(n-\#S)!}{\prod_{j\neq i}s_j!}\big[V_i\big], \end{aligned}$$

where $D=\dim H^{n-1}(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p}$, and so

$$n\binom{n}{\#S}\big[H(Y_{\mathrm{Iw}(m),S},\mathcal{L}_\xi)[\pi^p]^{U^p}\big] = (\dim H^{n-1}(X,\mathcal{L}_\xi)^{\mathrm{Iw}(m)}[\pi^p]^{U^p})\sum_{s_i=\#S}\big[V_i\big].$$

As $\Pi_{F,w}$ is tempered, $\mathrm{rec}\big(\Pi_{F,w}^\vee \otimes (\psi_u^\vee \circ \mathbf{N}_{F_w/E_u})|\det|^{\frac{1-n}{2}}\big)$ is pure of weight $w_\xi + (n-1)$. Hence

$$V_i = \mathrm{rec}\big(\pi_i^{-1}|\ |_w^{\frac{1-\#S}{2}} (\psi_u \circ \mathbf{N}_{F_w/E_u})^{-1}|\ |_w^{\frac{\#S-n}{2}}\big)$$

is strictly pure of weight $w_\xi + (n - \#S)$. The Weil conjectures then tell us that

$$H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)[\pi^p]^{U^p} = (0)$$

for $j \neq n - \#S$. The theorem follows. $\square$

We can now conclude the proof of theorem 1.1. Choose k so that $|\chi_\Pi| = |\ |_L^{\frac{n(k+n-1)}{2}}$ where χ_Π is the central character of Π. Set

$$V = H^{n-1}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)[\pi^p]^{U^p} \otimes R_l(\psi)^{-1},$$

a continuous representation of $\mathrm{Gal}\,(F^{ac}/F)$. We know that

(1) $V^{\mathrm{ss}} \cong R_l(\Pi)|^a_{\mathrm{Gal}\,(F^{ac}/F)}$ for some $a \in \mathbb{Z}_{>0}$,
(2) V is pure of weight k (proposition III.2.1 of [HT] and a computation of the determinant),
(3) $\mathrm{WD}(V|_{\mathrm{Gal}\,(F_w^{ac}/F_w)})$ is pure of weight k (use theorem 3.2 and a computation of the determinant).

Thus lemma 1.4 tells us that $\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{F\text{-ss}}$ is pure. On the other hand, as Π_v is tempered (corollary VII.1.11 of [HT]), $\mathrm{rec}\big(\Pi_v^\vee|\det|^{\frac{1-n}{2}}\big)$ is pure by lemma 1.3(3). As the representation of the Weil group in $\mathrm{rec}\big(\Pi_v^\vee|\det|^{\frac{1-n}{2}}\big)$ and $\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{F\text{-ss}}$ are equivalent, we deduce from lemma 1.3(4) that

$$\imath\mathrm{WD}(R_l(\Pi)|_{\mathrm{Gal}\,(L_v^{ac}/L_v)})^{F\text{-ss}} \cong \mathrm{rec}\big(\Pi_v^\vee|\det|^{\frac{1-n}{2}}\big),$$

as desired.

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